International Journal of Theoretical Physics, Vol. 40, No. 9, September 2001 (© 2001)

Generally Covariant Schrödinger Equation in Newton–Cartan Space–Time. Part I

J. Wawrzycki[1]

Received January 26, 2001

The covariant Schrödinger equation is obtained with the use of standard geometrical objects of the Galilean space–time. It's symmetry and covariance are investigated. Gauge freedom is eliminated by invariance condition. Family of the plane wave solutions in any coordinate system is found. Connection with previous investigations is discussed.

1. INTRODUCTION

Of late there has been a great interest in frame of reference rotating with a constant angular velocity (Białynicki-Birula *et al.*, 1994; Białynicki-Birula and Białynicki-Birula, 1997; Bordé *et al.*, 1991; Kaliński *et al.*, 1996; Lämmerzahl, 1996; Mashoon, 1988). The case of rotating motion with constant angular velocity is exceptional in the sense that the correct Schrödinger equation can be uniquely obtained from the classical Lagrange function (Białynicki-Birula *et al.*, 1994; Białynicki-Birula and Białynicki-Birula, 1997; Bordé *et al.*, 1991; Kaliński *et al.*, 1996; Lämmerzahl, 1996; Mashoon, 1988). But this is not the case in general for two reasons

(a) there is the problem with succession of operators;
(b) in general such operators appear that do not possesses any self-adjoint extension.

So, a natural problem arises: to write Schrödinger's equation in any noninertial coordinate system. Duval and Künzle (1984) and Kuchař (1980) gave such an equation. Kuchař gave explicit form of the equation for special observers, and in the case of rotating observers it has a form unitarily unequivalent to that given by Białynicki-Birula *et al.* (1994), Białynicki-Birula and Białynicki-Birula (1997), Bordé *et al.* (1991), Kaliński *et al.* (1996), Lämmerzahl (1996), and Mashoon

[1] Department of Theoretical Physics, Marian Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, 30-059 Kraków, Poland; e-mail: jwaw@thrisc.if.uj.edu.pl.

(1988) (see Eq. (6.11) and Eq. (4.17) of Kuchař, 1980). But the Eqs. (6.8) and (4.17) of his work are in agreement with these authors' work. Equation (6.11) written for *rigid* observers is incorrect. The Eq. (6.8) or (5.4) is essentially the same as the equation given by Duval and Künzle. Duval and Künzle (1984) found a covariance scheme of the Schrödinger equation in a very elegant way. They built the so-called *Bargmann principal bundle* $B(M)$ over the space–time M (it is built in a standard way, i.e. exactly as the principal bundle $G(M)$ with the Galilean group G, but instead of G the central extension B (*Bargmann group*) of G is used. Then, they found the most general connection form ω on it.

But their scheme possesses too big a gauge freedom, and the number of objects they use is also too big as considered in the Schrödinger's equation context, that is, *those objects cannot be defined in terms of Newton–Cartan space–time* (in the appendix, the works of Duval and Künzle (1984) and Kuchař (1980) are compared in detail).

Moreover, it seems that if we want to have the wave equation in generally covariant form a gauge freedom is needed (even in Galilean space–time). It is really the case if we take into account only the *covariance* properties (not *invariance*). In the literature concerning the problem only *covariance* is taken into account (not only in Duval and Künzle (1984) but also in the less known and quite recent literature (Canarutto *et al.*, 1995; Vitolo, 1999). On the other hand, no gauge freedom appears in description of a free quantum particle. This problem is solved here, i.e. it is shown in this paper that the gauge freedom can be uniquely eliminated by *invariance* properties.

There exists another important motivation for research in generally covariant wave equation: *the most general schemas of covariance provide a very natural way in which the wave function interacts with other fields.*

According to this idea, originated by Hermann Weyl (as to the author's knowledge), the interaction is realized by an appropriate definition of the connection. Application of the idea to the nonrelativistic case has at least two justifications. The first is that the nonrelativistic quantum mechanics is quite well understood and has mathematical support, so it is a guide to the relativistic case (see Canarutto *et al.*, 1995). The second justification is as follows: The *Galilei invariant* wave function in the electromagnetic field is still unknown, or the interaction of the wave function with the *Galilean* limit of the Maxwell equations as firstly found by Le Bellac and Lévy-Leblond (1973). As was shown in Le Bellac and Lévy-Leblond (1973) there are two different limits: electric and magnetic. The magnetic one is "more consistent" with the quantum mechanics in the sense that the wave function obtained by the minimal coupling is indeed covariant but in the electric case it is not. However, the constitutive equations are nonlinear in the magnetic case (G. A. Goldin, private communication). This needs an explanation.

The aforementioned problem does not contradict the agreement of the semiclassical theory of radiation with experiments. It is physically reasonable (and of course physically possible) to assume that in comparison to the light velocity, atom has small velocity when it interacts with the electromagnetic waves. After this the Hamilton function of the atom is classical but the electromagnetic waves fulfil relativistic equations, there do not exist electromagnetic waves in the Galilean limit of Maxwell equations (see Le Bellac and Lévy-Leblond, 1973).

In this paper, we have not dealt with the more general and most natural coupling to other fields, the reader interested in it may refer to Canarutto *et al.* (1995), where some interesting ideas from this field are presented. We deal with the coupling to the gravitational field and in part I mainly with the covariant wave equation in the flat Galilean space–time.

In Part I the generally covariant Schrödinger equation is given. It correctly reproduces results of Białynicki-Birula *et al.* (1994), Białynicki-Birula and Białynicki-Birula (1997), Bordé *et al.* (1991), Kaliński *et al.* (1996), Lämmerzahl (1996), and Mashoon (1988) in the case of rotating motion with constant angular velocity. The Schrödinger equation is obtained with the use of standard geometrical space–time objects only, introduced independently by Daŭtcourt (1990). It should be stressed that only the standard geometrical objects—and no other quantities—are used. The explicit form of the gauge, which brings the equation to the form *invariant* with respect to all space–time symmetries, is given. This establishes the *geometrical interpretation of the wave function* in Galilean space–time. The *phase* f transforming the wave function Ψ is uniquely determined by *invariance condition* (see section 5). Also, the generally covariant formulation of the classical mechanics is given and connection of the generally covariant Hamilton–Jacobi to the generally covariant wave equation. This gives possibility of finding the form of the plane wave in any coordinate system. All this would be very difficult to obtain with the use of the method applied in Canarutto *et al.* (1995) (see the comments under the number IV of the introduction in Canarutto *et al.*, 1995).

In Part II the generally covariant wave equation in the Newton–Cartan space–time is investigated.

Because the notions of *invariance* and *covariance* are important in our investigations, we give strict and general definitions of them. Let us consider a space–time M and the group G (or pseudogroup) of transformations of M (in our case the group of diffeomorfisms).

Definition. There is given a geometrical object $y(y)$ in M with m components. If for each point x there exists a neighborhood such that m numbers y correspond uniquely to each point of the neighborhood; the correspondence is such that the components y' at each point x in a new coordinate system u' depends only on the components y in the old system u and the transformation t of G, $t : u \to u'$, i.e.

$$y' = F(y, t), \qquad t : u \to u'.$$

See Schouten (1951) and Nijenhuis (1952) for the literature concerning definition and investigation of the geometric objects.

Let us consider a physical system that by assumption is completely described by an object y (in the case of a free particle y is the space–time curve—possibly its history). The set of all possible values of y (not necessarily realisable physically by the system) will be called to be the set of "kinematically possible trajectories" (kpt) (in the case of a freely falling particle the set of kpt consists of all space–time curves, however, not necessarily geodesic curves). If the y can be physically realized, then it will be called "dynamically possible trajectory" (dpt) (in the case of a free falling particle the set of (dpt) consists of all geodesics).

Definition. A group G will be called covariance group of a theory if

1. The set of all kpt constitutes a geometric object under the action of G.
2. The action in 1 is such that it associates dpt with dpt.

If a theory possesses a covariance group G, then one can divide the set of dpt into equivalence classes of a given dpt. Two dpt are defined to be members of the same class if they are associated by an element of G. Equivalence class represents the same physical state of a system, but in a different reference frame. In general it is possible to divide ys in two parts—dynamical yD and absolute yA—in such a way that:

Definition.

1. The part yD is that which distinguishes between various equivalence classes.
2. yD constitute a geometrical object under the action of G.
3. yA constitute a geometrical object under the action of G.
4. Any yA that satisfies the equations of motion of the theory appears, together with all its transforms under G, in every equivalence class of dpt.

Exceptionally, when there exists only one equivalence class, y is wholly absolute (especially theory describing flat Gallillean space–time is an example of this exceptional case, y denotes full geometrical description of that space–time).

Definition. The subgroup of the covariance group G, which is the symmetry (invariance) group of all absolute objects yA is said to be symmetry group of that theory.

The explicit definition of *invariance* is given in section 5.

In the case of the quantum theory of a free particle in Galilean space–time all geometrical quantities describing space–time are absolute objects, of course, and the inhomogeneous Galilean group is the symmetry group of that theory. In section 5 the explicit meaning of the *invariance* condition will be presented in this theory.

The motivation to the use of the notion of the *geometric object* is that in our case and in general case the quantities we are dealing with do not in general form any representation of G but they are always geometric objects. As an example, let us consider the wave function. It is well known that it does not form any representation of G but only a ray representation.

The paper is organized as follows. In section 2 geometrical structure of Galilean space–time is described following Daŭtcourt (1990). In section 3 generally covariant Hamilton–Jacobi equation is obtained with the help of Courant and Hilbert's theorem. In section 4 a generalization of Schrödinger Ansatz is given, by which Schrödinger (1926) passed from Hamilton–Jacobi equation originally to the Schrödinger equation. In section 5 Galilean *invariance* of the equation will be proved. In section 6 it is shown that the equation correctly reconstructs results of Białynicki-Birula *et al.* (1994), Białynicki-Birula and Białynicki-Birula (1997), Bordé *et al.* (1991), Kaliński *et al.* (1996), Lämmerzahl (1996), and Mashoon (1988). In the appendix, connection with the work of Duval and Künzle (1984) and Kuchař (1980) is presented.

2. NEWTON–CARTAN SPACE–TIME

Contrary to general relativity space–time the Newton–Cartan space–time (especially the Galilean space–time) is described by three independent geometrical objects: the connection $\Gamma^{\mu}_{\nu\rho}$, the gradient of absolute time t_{μ}, and contravariant tensor field $g^{\mu\nu}$, with the rank equal to 3: $g^{\mu\nu}t_{\nu} = 0$. See Trautman (1963) for a more detailed discussion. They are covariantly constant: $\nabla_{\mu}g^{\lambda\rho} = 0$ and $\nabla_{\mu}t_{\nu} = 0$. But because the rank of $g^{\mu\nu}(3\Gamma^{\mu}_{\nu\rho})$ is not determined by $g_{\mu\nu}$ and t_{μ}. With the help of $\Gamma^{\mu}_{\nu\rho}$, however, covariant tensor $g_{\mu\nu}$ and contravariant vector $\boldsymbol{u}$ can be defined such that

$$g^{\lambda\nu}g_{\nu\beta}g^{\beta\mu} = g^{\lambda\mu}. \tag{1}$$

All the rest of that paragraph will concern the precise definition of covariant g (and an additional object—contravariant vector $\boldsymbol{u}$, see (2) and (3)). Equation (1) is not a definition, of course. The covariant g and contravariant $\boldsymbol{u}$ completely replace the connection and are much more convenient herein than in the other connection. $\Gamma^{\mu}_{\nu\rho}$ is determined by motions of free-falling particles, i.e. geodesics. Geodesic is a solution of the Lagrange–Euler equations for a free-particle Lagrange function L. The action S of the particle in a Cartesian coordinate system (t, X, Y, Z) has the well-known form

$$S = \int \frac{m}{2}\left\{\frac{dX^2 + dY^2 + dZ^2}{dt^2} - 2\varphi\right\} dt$$

with the Newtonian potential φ. If one introduces a new parameter τ along the space–time curve of a particle instead of absolute time t, the action will take on the form

$$\int \frac{m}{2} \frac{\left(\frac{dX}{d\tau}\right)^2 + \left(\frac{dY}{d\tau}\right)^2 + \left(\frac{dZ}{d\tau}\right)^2 - 2\varphi\left(\frac{dt}{d\tau}\right)^2}{\frac{dt}{d\tau}}\, d\tau.$$

Then we write the last expression in arbitrary coordinates (x^μ, $\mu = 0, 1, 2, 3$) (keeping the same parameter along the space–time curve) defined as functions of the Cartesian coordinates (t, X, Y, Z):

$$\int \frac{m}{2} \frac{a_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu}{t_\sigma\, \dot{x}^\sigma} d\tau,$$

where dot denotes differentiation with respect to the parameter τ and $t_\sigma \equiv \frac{\partial t}{\partial x^\sigma}$ (hereafter differentiation will be denoted by ∂_σ). Lagrange function L becomes homogeneous of degree 1 in the velocities $\dot{\xi}^\mu$, and can be written as

$$L = \frac{m}{2} \frac{a_{\mu\nu}\, \dot{\xi}^\mu \dot{\xi}^\nu}{t_\sigma \dot{\xi}^\sigma},$$

with some covariant field $a_{\mu\nu}$. The parameter $g_{\mu\nu}$ can be interpreted as equal to $a_{\mu\nu}$, that is

$$L = \frac{m}{2} \frac{g_{\mu\nu}\, \dot{\xi}^\mu \dot{\xi}^\nu}{t_\sigma \dot{\xi}^\sigma},$$

and because L is determined up to full-parameter derivative

$$L \to L + m\, \frac{df}{d\tau},$$

$g_{\mu\nu}$ is determined up to the gauge transformation

$$g_{\mu\nu} \to g_{\mu\nu} + t_\mu\, \partial_\nu f + t_\nu\, \partial_\mu f.$$

It is easy to check that $g_{\mu\nu}$ defined in this way fulfils (1). As a second step $\boldsymbol{u}^\mu$ is defined in the following way

$$\begin{aligned} g_{\mu\nu} g^{\nu\sigma} &= \delta^\sigma_\mu - \boldsymbol{u}^\sigma t_\mu \\ \boldsymbol{u}^\mu t_\mu &= 1. \end{aligned} \tag{2}$$

It is defined, of course, up to the gauge transformation

$$\boldsymbol{u}^\mu \to \boldsymbol{u}^\mu - g^{\mu\nu}\, \partial_\nu f.$$

Lagrange–Euler equations give the geodesic equation with the connection

$$\Gamma^\mu_{\nu\rho} = \boldsymbol{u}^\mu \partial_\nu t_\rho + \frac{1}{2} g^{\mu\sigma} \{\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\} \tag{3}$$

as a gague-independent quantity. The last formula was proved independently by Daŭtcourt (1990). Conversely, (3) and (2) up to the gauge determine covariant g and contravariant $\boldsymbol{u}$, so they may be introduced instead of the connection, as will be done here. It should be stressed here that the covariant g and contravariant $\boldsymbol{u}$ completely replaces the affine connection.

3. GENERALLY COVARIANT HAMILTON–JACOBI EQUATION

If the Lagrange function L is homogeneous of degree 1 in velocities the Hamilton–Jacobi equation can also be built up. In this particular case we have $\det(\partial_{\dot\xi^\nu}\partial_{\dot\xi^\nu} L) = 0$ and $H = -L + \dot\xi^\mu \partial_{\dot\xi^\mu} L = 0$ and the canonical form cannot be reached by the Legendre transformation. In this case, however, the following equations remain valid (see e.g., Courant and Hilbert, 1961).

$$\partial_{\xi^\mu} S = \partial_{\dot\xi^\mu} L$$

with the principal Hamilton–Jacobi function S. Homogeneity relation $\dot\xi^\mu \partial_{\dot\xi^\mu} L = L$ with $\partial_{\xi_\mu} S$ substituted instead of $\partial_{\dot\xi^\mu} L$ is the Hamilton–Jacobi equation

$$\boldsymbol{u}^\mu \partial_{\xi^\mu} S + \frac{1}{2m} g^{\mu\nu} \partial_{\xi^\mu} S \, \partial_{\xi^\nu} S - \frac{m}{2} g_{\mu\nu} \boldsymbol{u}^\mu \boldsymbol{u}^\nu = 0.$$

This is a particular case of the theorem of Courant and Hilbert for homogeneous Lagrange function L. The Hamilton–Jacobi equation uniquely follows from this theorem. With the help of this equation transformation properties of S can be investigated. From this equation, it follows that under the coordinate transformation S transforms as a scalar field and under the gauge transformation S has the following transformation rule

$$S \to S + mf. \tag{4}$$

The phase of the plane wave function describing a free particle is exactly equal to S. Therefore from (4) transformation rule of the wave function Ψ follows. Under the gauge transformation f namely, the transformation is

$$\Psi \to e^{\frac{im}{\hbar} f} \Psi$$

and under a coordinate transformation

$$\Psi(\xi^\mu) \to \Psi'(\xi^{\mu'}) = \Psi(\xi^\mu(\xi^{\mu'})).$$

4. SCHRÖDINGER'S ANSATZ AND THE EXPLICIT FORM OF THE PLANE WAVE IN ANY COORDINATE SYSTEM

The covariant Schrödinger equation may be found with the help of slightly modified Schrödinger Ansatz (see Schrödinger, 1926) using the covariant

Hamilton–Jacobi equation. This derivation cannot be regarded as a proof. The proof consists of the investigations of the *invariance* properties of the wave equation presented in section 5. At the beginning a small modification of the Ansatz that led to Schrödinger's equation *without time* will be presented. The modified Ansatz leads to Schrödinger's equation *with time* (the original Schrödinger's derivation of wave equation *with time* is different, the presented Ansatz is based on Ansatz of the equation *without time*). The starting point of Schrödinger's considerations was de Broglie's hypothesis. He used the analogy

$$\begin{array}{ccc} \text{de Broglie's wave} & \longleftarrow\!\!\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!\!\!\longrightarrow & \text{electromagnetic wave} \\ \text{classical limit} \downarrow & & \downarrow \text{geometrical optics limit} \\ \text{classical particle} & \longleftarrow\!\!\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!\!\!\longrightarrow & \text{light ray} \end{array}$$

Eikonal is the counterpart of the classical action S. Fermat's principle is the counterpart of principle of least action. From this he derived the form of a wave function Ψ

$$\Psi = a e^{\frac{i}{\hbar} S},$$

where a is a constant, which is cancelled by differentiation in the next steps. So, this may be formally written as

$$S = \frac{\hbar}{i} \ln \Psi.$$

Schrödinger substituted this S to the Hamilton–Jacobi equation

$$\partial_t S + \frac{1}{2m} \vec{\nabla} S \cdot \vec{\nabla} S = 0,$$

where the positive real number $\vec{\nabla} S \cdot \vec{\nabla} S$ and real number $\partial_t S$ was written as $\hbar^2 \frac{\vec{\nabla}\Psi}{\Psi} \cdot \frac{\vec{\nabla}\Psi^*}{\Psi^*}$ and $\frac{\hbar}{2i}(\frac{\partial_t \Psi}{\Psi} - \frac{\partial_t \Psi^*}{\Psi^*})$, respectively:

$$-\frac{\hbar^2}{2m} \vec{\nabla}\Psi \cdot \vec{\nabla}\Psi^* + \frac{i\hbar}{2}(\Psi^* \, \partial_t \Psi - \Psi \, \partial_t \Psi^*) = 0. \tag{5}$$

However Schrödinger did not solve it. He probably thinks on the membrane vibration theory and assumes that the wave function Ψ minimizes a functional

$$\int\!\!\int_{R^3} \ell(\Psi) \, dt \, d^3x$$

quadratic in Ψ and its derivatives. He assumes that the Lagrange function $\ell(\Psi)$ is equal to the left-hand side of (5). So, the Schrödinger equation follows from

$$\delta\left(\int_{t_1}^{t_2} \int_{R^3} \ell(\Psi) \, dt \, d^3x\right) = 0,$$

where $\delta\Psi(t_1) = \delta\Psi(t_2) = 0$. To have the functional $\int_{t_1}^{t_2} \int_{R^3} \ell(\Psi)\, dt\, d^3x$ well defined, he assumes that Ψ tends to zero appropriately fast when x goes to infinity, so that

$$\int_{R^3} \Psi\Psi^*\, d^3x = 1.$$

Now the generally covariant Schrödinger equation will be derived. The analogous substitutions are

$$S = \frac{\hbar}{i} \ln \Psi,$$

$$\boldsymbol{u}^\mu\, \partial_\mu S = \frac{\hbar}{2i}\boldsymbol{u}^\mu \left(\frac{\nabla_\mu \Psi}{\Psi} - \frac{\nabla_\mu \Psi^*}{\Psi^*}\right)$$

and

$$g^{\mu\nu}\, \partial_\mu S\, \partial_\nu S = \hbar^2 g^{\mu\nu} \frac{\nabla_\mu \Psi}{\Psi}\frac{\nabla_\nu \Psi^*}{\Psi^*}.$$

Exactly as before these formulas are substituted to the generally covariant Hamilton–Jacobi equation. The result of the substitution is as follows

$$\frac{i\hbar}{2}\boldsymbol{u}^\mu(\Psi^*\nabla_\mu\Psi - \Psi\nabla_\mu\Psi^*) - \frac{\hbar^2}{2m} g^{\mu\nu}\nabla_\mu\Psi\nabla_\nu\Psi^* + \frac{m}{2} g_{\mu\nu}\boldsymbol{u}^\mu\boldsymbol{u}^\nu\Psi\Psi^* = 0.$$

The left-hand side of the last formula will be denoted by $\Lambda(\Psi)$. Following Schrödinger the Lagrange function ℓ of generally covariant Schrödinger equation will be defined as equal to $\Lambda(\Psi)\cdot v$, where v is the *natural invariant measure* of Galilean space–time, that is, the wave equation follows from $\delta(\int \Lambda v\, d^4x) = 0$. Now it will be shown that this condition is equivalent to

$$\frac{\partial\Lambda}{\partial\Psi^*} - \nabla_\mu \frac{\partial\Lambda}{\partial(\nabla_\mu\Psi^*)} = 0. \tag{6}$$

The simplest way to compute the *natural invariant measure* v is to pass to such a coordinate system that has as one coordinate the absolute time t, and remaining three coordinates x as any space coordinates entirely lying in a simultaneity hyperplane. In such coordinates the volume element $v\, d^4x$ is of the form $\sqrt{g}\, dt\, d^3x$, where g denotes the determinant of Euclidean metric tensor matrix on a simultaneity hyperplane. It can be shown the Euclidean tensor is induced by contravariant tensor g. The principle of least action is equivalent to

$$\begin{aligned} 0 &= \frac{\partial\ell}{\partial\Psi^*} - \partial_\mu \frac{\partial\ell}{\partial(\partial_\mu\Psi^*)} = \sqrt{g}\,\frac{\partial\Lambda}{\partial\Psi^*} - \sqrt{g}\,\partial_{x^a} \frac{\partial\Lambda}{\partial(\partial_{x^a}\Psi^*)} - \sqrt{g}\partial_t \frac{\partial\Lambda}{\partial(\partial_t\Psi^*)} \\ &\quad -\partial_{x^a}\sqrt{g}\,\frac{\partial\Lambda}{\partial(\partial_{x^a}\Psi^*)} - \partial_t\sqrt{g}\,\frac{\partial\Lambda}{\partial(\partial_t\Psi^*)} = \sqrt{g}\left(\frac{\partial\Lambda}{\partial\Psi^*} - \nabla_\mu \frac{\partial\Lambda}{\partial(\nabla_\mu\Psi^*)}\right), \end{aligned}$$

where the explicit form of affine connection in the coordinates (t, x) has been used (see Daŭtcourt, 1990): $\partial_t \sqrt{g} = \sqrt{g}\frac{1}{2}g^{ik}\, \partial_t g_{ik} = \sqrt{g}\Gamma^a_{oa}$, $\partial_{x^a}\sqrt{g} = \sqrt{g}\Gamma^b_{ab}$, $\Gamma^o_{\nu\rho} = 0$ and Γ^c_{ab} is equal to ordinary Christoffel symbol of the Euclidean tensor g^{ab} (time coordinate is denoted by 0, space coordinates are denoted by Latin indices). Ψ and Ψ^* have been considered as independent functional variables. Because the left-hand side of (6) is a scalar vanishing in (t, x) frame, it is equal to zero in any coordinate system. Conversely, because Lagrange derivative of ℓ is a density of weight $+1$ (which (from (6)) vanishes in (t, x) frame) it is equal to zero in any frame.

As to the *invariant measure* in any frame of reference one has

Theorem. *In any coordinate system* (x^μ) *invariant measure* v *is*

$$v \equiv \sqrt{det[g_{\mu\nu} + (1 - g_{\alpha\beta}\boldsymbol{u}^\alpha \boldsymbol{u}^\beta)t_\mu t_\nu]}.$$

Proof: First, gauge-independent covariant tensor $M_{\mu\nu}$ can be defined:

$$M_{\mu\nu} = g_{\mu\nu} - g_{\mu\lambda}\boldsymbol{U}^\lambda t_\nu - g_{\nu\lambda}\boldsymbol{U}^\lambda t_\mu + g_{\alpha\beta}\boldsymbol{U}^\alpha \boldsymbol{U}^\beta t_\mu t_\nu + t_\mu t_\nu,$$

where $\boldsymbol{U}^\lambda$ is any vector, such that $\boldsymbol{U}^\lambda t_\lambda = 1$. Tensor $M_{\lambda\mu}$ possesses the inverse $M^{\mu\nu}$:

$$M^{\mu\nu} = g^{\mu\nu} + \boldsymbol{U}^\mu \boldsymbol{U}^\nu,$$

that is, determinant of $M_{\mu\nu}$ is not equal to zero and

$$M_{\mu\nu} M^{\nu\sigma} = \delta^\sigma_\mu,$$

if $\boldsymbol{U}^\mu = \boldsymbol{u}^\mu$. Second, determinant of $M_{\mu\nu}$ *does not depend on* U^μ fulfilling the condition $U^\mu t_\mu = 1$. Indeed, one has

$$[\partial_{\boldsymbol{U}^\mu} M_{\mu\nu} M^{\mu\nu}]_{\boldsymbol{U}^\lambda t_\lambda = 1} = 0,$$

and

$$[\partial_{\boldsymbol{U}^\beta} \det(M_{\mu\nu})]_{\boldsymbol{U}^\lambda t_\lambda = 1} = [\partial_{\boldsymbol{U}^\beta} M_{\mu\nu} M^{\mu\nu} \det(M_{\rho\sigma})]_{\boldsymbol{U}^\lambda t_\lambda = 1} = 0.$$

The substitution $\boldsymbol{U}^\mu \stackrel{df}{=} \boldsymbol{u}^\mu$ gives $v \stackrel{df}{=} \sqrt{\det(M_{\mu\nu}(\boldsymbol{U}^\lambda = \boldsymbol{u}^\lambda)}$ mentioned earlier. As the last step of the proof it will be shown that $v = \sqrt{g}$ in the coordinates (t, x), where t is the absolute time and x are the remaining three coordinates lying in simultaneity hyperplane (g is defined as earlier). Substituting the explicit form of $g_{\mu\nu}$ and $\boldsymbol{u}^\mu$ in (t, x) (see Daŭtcourt, 1990) one has

$$\det(M_{\mu\nu}) = \det\begin{pmatrix} 1 + g^{ab}g_{oa}g_{ob} & g_{ob} \\ g_{oa} & g_{ab} \end{pmatrix} = g + gg^{ab}g_{oa}g_{ob} - gg^{ab}g_{oa}g_{ob} = g.$$

□

Because $\sqrt{\det(M_{\mu\nu})}$ is a scalar density of weight +1 equal to *invariant measure* in (t, x), it is equal to *invariant measure* in any coordinate system (Eq. (6) may be derived independently with the help of explicit form of v).

From (6) one gets the generally covariant Schrödinger equation:

$$i\hbar \boldsymbol{u}^{\mu}\partial_{\mu}\Psi = -\frac{\hbar^2}{2m}g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\Psi - \frac{m}{2}g_{\mu\nu}\boldsymbol{u}^{\mu}\boldsymbol{u}^{\nu}\Psi - \frac{i\hbar}{2}\nabla_{\mu}\boldsymbol{u}^{\mu}\Psi.$$

It is evidently covariant with respect to coordinate transformations. It is covariant with respect to gauge transformations too and unitarily equivalent to ordinary Schrödinger equation in an inertial frame of reference. Moreover, the relation of the generally covariant Hamilton–Jacobi equation to generally covariant Schrödinger equation is completely analogous to the relation of ordinary Hamilton–Jacobi equation to ordinary Schrödinger equation. For example, the plane wave in any frame of reference with given canonical momentum is a solution of generally covariant wave equation if and only if the momentum is equal to the gradient of a solution of generally covariant Hamilton–Jacobi equation.

Now the form of a plane wave solution in any co-ordinate system can be found. Canonical momentum $p_{\mu} = \frac{\partial L}{\partial \dot{x}^{\mu}}$ is a gauge-dependent covector and the gauge f transforms it in the following way

$$p_{\mu} \to p_{\mu} + m\partial_{u} f.$$

It is necessary to compute canonical momentum in terms of a gauge-independent quantity to find the plane wave solution, namely the fourvelocity of described particle. Two momentas describe the same particle in two coordinate systems if they are connected with the same fourvelocity. Such a definition is correct because fourvelocity is gauge-independent quantity.

It is easy to show that

$$g^{\mu\nu}p_{\nu} + m\boldsymbol{u}^{\mu} = m\boldsymbol{V}^{\mu},$$

where $\boldsymbol{V}^{\mu}$ is the fourvelocity. Then with the help of the covariant Hamilton–Jacobi equation and the fact that $\boldsymbol{V}^{\mu}t_{\mu} = 1$ one can compute

$$p_{\lambda} = mg_{\lambda\mu}\boldsymbol{V}^{\mu} - \frac{m}{2}g_{\mu\nu}\boldsymbol{V}^{\mu}\boldsymbol{V}^{\nu}t_{\lambda}.$$

With a free-falling particle in a flat space–time the covariantly constant fourvector field can be connected in a natural way such, that $\nabla_{\lambda}\boldsymbol{V}^{\mu} = 0$ and $\boldsymbol{V}^{\mu}t_{\mu} = 1$. Then the plane wave solution with the momentum p_{μ} is of the form

$$\Psi_{p_{\mu}}(p) = e^{\frac{i}{\hbar}\int_{po}^{p}\partial_{\mu}S\,dx^{\mu}} \equiv e^{\frac{i}{\hbar}\int_{po}^{p}p_{\mu}\,dx^{\mu}}.$$

The exponent is understood as a curvilinear integral. First, it should be stressed that it is well defined because under the gauge transformation momentum transforms by δp_μ, the gradient of f, and $\oint \delta p_\mu \, dx^\mu = 0$. Second, the plane wave defined in this way fulfils the following five conditions

1. It is a solution of the Schrödinger equation.
2. Under the coordinate transformation it behaves like a scalar.
3. Under the gauge transformation f it transforms like
$$\Psi_{p_\mu} \to e^{\frac{im}{\hbar} f} \Psi_{p_\mu}.$$
4. A gauge can be chosen such that the quantities $g_{\mu\nu}$ and $\boldsymbol{u}^\mu$ are *invariant* with respect to Galilean transformations (the gauge will be called *symmetric*).[2] The plane wave solution has ordinary plane wave form in an inertial reference frame when the gauge is *symmetric*.
5. If one performs the transformation $x^\mu \to x^{\mu'}$ and gauge f the plane wave with momentum p_μ goes into the plane wave with the momentum
$$p'_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} (p_\mu + m \, \partial_\mu f).$$

It should be stressed that there does not exist any plane wave solution when the gravity field is present and the Riemann curvature is not zero.

5. THE EXPLICIT FORM OF THE GAUGE IN WHICH THE WAVE EQUATION IS INVARIANT

Combining in an appropriate way the gauge transformation and coordinate transformation one may bring in the situation when $g_{\mu\nu}$ and $\boldsymbol{u}^\mu$ are *invariant* with respect to Galilean transformations. However, the condition of invariance is not equivalent to putting the ordinary Lie derivative to zero. This is because the quantities are not simply tensors when their transformation is combined with appropriate gauge f (called *symmetric*)[3]

$$\begin{aligned} g_{\mu\nu} &\to \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} (g_{\mu\nu} + t_\mu \, \partial_\nu f + t_\nu \, \partial_\mu f) \\ \boldsymbol{u}^\mu &\to \frac{\partial x^{\mu'}}{\partial x^\mu} (\boldsymbol{u}^\mu - g^{\mu\nu} \, \partial_\nu f). \end{aligned} \tag{7}$$

At the beginning the notion of the *invariance* of any object (whatever transformation rule it has) will be given according to Schouten (1951). Let the points of a region R of space–time be subject to the point transformation

$$\eta^\kappa = f^\kappa(\xi^\lambda). \tag{8}$$

[2] See the next paragraph for more detailed discussion of the *invariance and symmetric gauge*.

[3] Strictly speaking $g_{\mu\nu}$ composes a geometric object after the above defined choice of the phase f but $\boldsymbol{u}^\mu$ does not. However, $\boldsymbol{u}^\mu$ together with $g_{\mu\nu}$ composes a geometric object.

The functions f^κ are supposed to be analytic in R with a nonvanishing functional determinant

$$\left|\frac{\partial f^\kappa}{\partial \xi^\lambda}\right| \neq 0,$$

and to be chosen in such a way that they determine a one-to-one correspondence between the points of R and the points of another region R'. Now we introduce another coordinate system (κ') such that each point in R has the same coordinates with respect to (κ) as its image in R' has with respect to (κ'). The $\xi^{\kappa'}$ must be equal to the coordinates with respect to (κ) of the corresponding point of R. Hence, if

$$\xi^\kappa = \phi^\kappa(\eta^\lambda)$$

is the inversion of (8), the $\xi^{\kappa'}$ must be equal to the $\phi^\kappa(\xi^\lambda$ and accordingly the transformation of (κ) into (κ') and vice versa is given by the equations

$$\xi^{\kappa'} = \delta^{\kappa'}_\kappa \phi^\kappa(\xi^\lambda), \qquad \xi^\kappa = f^k(\xi^{\lambda'}).$$

This process will be called the *dragging along of the coordinate system* (κ) *by the point transformation* (*8*).

Now let some field S (eventual indices are omitted), be given in R. Let $\hat{S}$ be a second field in R', whose components with respect to (κ') in any point of R' are equal to the components of the first field S in the corresponding point of R. This process will be called the *dragging along of the field S by the point transformation* (*8*) and $\hat{S}$ will be called the *field dragged along*. If R and R' have some region in common, the fields S and $\hat{S}$ can be compared. If then $S = \hat{S}$, the field S is called *invariant for the point transformation* (*8*).

Substituting an infinitesimal transformation (8) and a tensor field for S to this definition one gets that ordinary Lie derivative is equal to zero. But substituting $g_{\mu\nu}$ and $\boldsymbol{u}^\mu$ with their transformation laws one gets

$$\begin{gathered}\partial_\mu(\delta\xi^\rho)g_{\rho\nu} + \partial_\nu(\delta\xi^\rho)g_{\mu\rho} + \delta\xi^\rho\,\partial_\rho g_{\mu\nu} = t_\mu\,\partial_\nu f(\delta\xi) + t_\nu\,\partial_\mu f(\delta\xi)\\ \partial_\nu(\delta\xi^\mu)\boldsymbol{u}^\nu - \delta\xi^\nu\,\partial_\nu \boldsymbol{u}^\mu = g^{\mu\nu}\,\partial_\nu f(\delta\xi),\end{gathered} \tag{9}$$

respectively, where f is the *symmetric phase of the Galilei transformations*, and the infinitesimal Galilei coordinate transformation was substituted

$$\xi^\mu \to \xi^\mu + \delta\xi^\mu.$$

Symmetric phase of the Galilean transformation (*Galilean symmetry phase*) in any coordinate system can be computed with the help of the plane wave solutions. Let (μ) be any coordinate system and let a geodesic of a free particle moving with a fourvelocity $\boldsymbol{V}^\mu$ be given ($\boldsymbol{V}^\mu$ is the tangential vector of the geodesic). Now let the space–time points be subject to the Galilean point transformation (not understood as a coordinate transformation but as a point transformation). The transformation moves the lines of (μ) coordinate system into (μ') lines, and the first geodesic into a second one with the tangential vector $\boldsymbol{V}'^{\mu} = \boldsymbol{V}^{\mu} + \boldsymbol{v}^{\mu}$, where $\boldsymbol{v}^{\mu}$ is defined as a *fourvelocity of the Galilean transformation*. Of course

$$\boldsymbol{v}^{\mu} t_{\mu} = 0 \tag{10}$$

because

$$1 = \boldsymbol{V}^{\mu} t_{\mu} = \boldsymbol{V}'^{\mu} t_{\mu}.$$

Again in a very natural way the covariantly constant fourvector field may be connected with the Galilean transformation (in a flat space–time) fulfilling (10) i.e. entirely lying in a symultaneity hyperplane. Wave functions of the first and the second particle are plane waves (built with the help of their fourvelocity fields, respectively; see preceding paragraph). Let the wave phase of the first particle in (μ) be S and the wave phase of the second one in (μ') be S'. The Galilean transformation is a symmetry if and only if

$$S(x^{\mu}, \boldsymbol{V}^{\lambda}) = S'(x^{\mu'}, \boldsymbol{V}'^{\lambda'}),$$

this is equivalent to

$$S(x^{\mu}, \boldsymbol{V}^{\lambda} - \boldsymbol{v}^{\lambda}) = S'(x^{\mu'}, \boldsymbol{V}^{\lambda'}).$$

With the help of this symmetry condition the quantity

$$\partial_{\nu'} S'(x^{\mu'}, \boldsymbol{V}^{\lambda'})\, dx^{\nu'}$$

may be computed, which gives the following theorem:

Theorem. *Gradient of the symmetric Galilean phase is equal:*

$$\partial_{\mu} f = -g_{\mu\nu}\boldsymbol{v}^{\nu} + \frac{1}{2} g_{\lambda\nu}\boldsymbol{v}^{\lambda}\boldsymbol{v}^{\nu} t_{\mu},$$

where $\boldsymbol{v}^{\mu}$ is the velocity field of the Galilean transformation $x^{\mu} \to x^{\mu} - \boldsymbol{v}^{\mu}\, dt$.

From (2) and (9) with Galilean transformations substituted, *symmetric Galilean phase* instead of f $g_{\mu\nu}$ and $\boldsymbol{u}^{\mu}$ can be computed. Moreover (2) and (9) are in agreement with (3), which is a very nontrivial fact. It can be shown that in the *symmetric gauge* the quantity

$$\eta \equiv -\frac{m}{2} g_{\mu\nu}\boldsymbol{u}^{\mu}\boldsymbol{u}^{\nu}$$

is a constant scalar called *internal energy* (connected with the unitary representations of the Galilei group; see the following discussion). Solving (2) and (9) in an inertial frame one gets

$$(g_{\mu\nu}) = \begin{pmatrix} -\frac{2\eta}{m} & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}, \qquad (\boldsymbol{u}^{\mu}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and the wave equation is

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}\vec{\nabla}^2\Psi + \eta\Psi.$$

η is any constant, which cannot be computed with the use of symmetry, because this equation is invariant with any constant value of η. The simplest way to explain it is to stress that in the phase space not only one paraboloid with $\eta = 0$ but the whole family is invariant with respect to Galilei group

$$E - \frac{\vec{p}^2}{2m} = \eta.$$

However, all unitary representations of the Galilei group with different values of η are equivalent (see Lévy-Leblond, 1963).

Solving (2) and (9) in a rotating frame (choosing $\eta = 0$) one gets the equation commonly applied when the rotating frame is used (see e.g., Białynicki-Birula *et al.*, 1994; Białynicki-Birula and Białynicki-Birula, 1997; Bordé *et al.*, 1991; Kaliński *et al.*, 1996; Lämmerzahl, 1996; Mashoon, 1988).

So, the *symmetric gauge* is found, but in a rather involved form. It is necessary to give the explicit form of the *symmetric phase f* for any transformation (not only Galilean) in any coordinate system. Again the problem will be solved using the plane wave solutions.

Let a free-falling particle geodesic be given, with a tangential fourvelocity $\boldsymbol{V}^{\mu}$ and its rest inertial reference frame (μ') (in the frame the particle is at rest). In addition let any reference frame (μ) be given (not necessarily inertial, curvilinear in general case). Suppose that $g_{\mu\nu}$ and $\boldsymbol{u}^{\mu}$ are *invariant*, and all used transformation phase f are supposed to be *symmetric*. Then in the rest frame the plane wave phase of the considered particle S' in (μ') is zero, such that

$$\partial_{\mu} S' = 0.$$

On the other hand gradient of the wave phase of the same particle in (μ) is

$$\partial_{\mu} S = m g_{\mu\nu} \boldsymbol{V}^{\nu} - \frac{m}{2} g_{\alpha\beta} \boldsymbol{V}^{\alpha} \boldsymbol{V}^{\beta} t_{\mu}.$$

Consider the transformation $(\mu) \rightarrow (\mu')$. It transforms the phase of the wave function $S \to S' = S + mf$. After differentiation one gets

$$m\,\partial_\mu f = -\partial_\mu S = -m g_{\mu\nu} \boldsymbol{V}^\nu + \frac{m}{2} g_{\alpha\beta} \boldsymbol{V}^\alpha \boldsymbol{V}^\beta t_\mu.$$

With the help of this f one may compute

$$g'_{\mu\lambda} = g_{\mu\lambda} + t_\mu\,\partial_\lambda f + t_\lambda\,\partial_\mu f = g_{\mu\lambda} - t_\lambda g_{\mu\nu} \boldsymbol{V}^\nu - t_\mu g_{\lambda\nu} \boldsymbol{V}^\nu + g_{\alpha\beta} \boldsymbol{V}^\alpha \boldsymbol{V}^\beta t_\mu t_\lambda.$$

Consider now $g'_{\mu\nu}$ near a space–time point p. $\boldsymbol{V}^\nu$ will be chosen such that $\boldsymbol{V}^\mu(p) = \boldsymbol{u}^\mu(p)$. Near p one has

$$\boldsymbol{V}^\mu = \boldsymbol{u}^\mu + (\boldsymbol{V}^\mu - \boldsymbol{u}^\mu) \equiv \boldsymbol{u}^\mu + \delta\boldsymbol{u}^\mu,$$

where $\delta\boldsymbol{u}^\mu(p) = 0$ and near p

$$g'_{\mu\lambda} = g_{\mu\lambda} - t_\lambda g_{\mu\nu}\,\delta\boldsymbol{u}^\nu - t_\mu g_{\lambda\nu}\,\delta\boldsymbol{u}^\nu + \text{ higher powers of } \delta\boldsymbol{u},$$

where η is chosen to be equal to zero. So

$$g'_{\mu\nu}(p) = g_{\mu\nu}(p);$$

in the case when η is not equal to zero the last formula is also true (in the case when $\eta \neq 0$, $\partial_\mu S' = -\eta t_\mu$ and

$$\partial_\mu f = -g_{\mu\nu} \boldsymbol{V}^\nu + \frac{1}{2} g_{\alpha\beta} \boldsymbol{V}^\alpha \left(\boldsymbol{V}^\beta t_\mu - \frac{\eta}{m} t_\mu \right).$$

This means that there exists an inertial frame moving with a fourvelocity $\boldsymbol{V}^\mu = \boldsymbol{u}^\mu(p)$ such that $g'_{\mu\nu}$ in it is equal to $g_{\mu\nu}$ in (μ) at the point p. So, if $\boldsymbol{u}^\mu$ was covariantly constant, $g_{\mu\nu}$ would be such as in an inertial reference frame (moving with the fourvelocity $\boldsymbol{V}^\mu = \boldsymbol{u}^\mu$). But from space–time translation invariance it follows that $\nabla_\mu \boldsymbol{u}^\nu = 0$, and one has the following theorem:

Theorem. *Symmetric phase f of any transformation* $(x^\lambda, \boldsymbol{u}^\mu) \to (x^{\lambda'}, \boldsymbol{u}'^{\mu'})$ *in any coordinate system always has the Galilean form*

$$\partial_\mu f = -g_{\mu\nu} \boldsymbol{v}^\nu + \frac{1}{2} g_{\lambda\nu} \boldsymbol{v}^\lambda \boldsymbol{v}^\nu t_\mu,$$

where $\boldsymbol{v}^\mu = \boldsymbol{u}^\mu - \boldsymbol{u}'^\mu$.

Symmetric phase f may be deduced independently from translation *invariance* of contravariant $\boldsymbol{u}$ and covariant g.

It is not obvious that Equations (9) are generally covariant with respect to (7); it can be shown that it is really the case when f is equal to *symmetric phase*. That is, if $\boldsymbol{u}$ and g are invariant in some reference frame, they will be *invariant* in any reference frame. Because our equation (uniquely determined by *invariance* conditions) is *invariant* in Galilean frame, it is invariant in any coordinate system.

6. INVARIANT FORM OF SCHRÖDINGER'S EQUATION IN A ROTATING FRAME

Frame of reference $(\vec{x}, t)$ rotating with a constant angular velocity ω around the z' axis of an inertial reference frame $(\vec{x}', t)$ will be considered here. *Velocity field* of a Galilean transformation in the rotating frame is of the form

$$(\boldsymbol{v}^\mu) = \begin{pmatrix} 0 \\ V^1 \cos\omega t - V^2 \sin\omega t \\ V^2 \cos\omega t + V^1 \sin\omega t \\ V^3 \end{pmatrix},$$

where $\boldsymbol{V}^a$ have been chosen in such a way that the Galilei transformation in rotating frame

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} t \\ x + V^1 t \cos\omega t - V^2 t \sin\omega t \\ y + V^1 t \sin\omega t + V^2 t \cos\omega t \\ z + V^3 t \end{pmatrix}$$

is equal to ordinary inertial form of Galilean transformation with the velocity $\boldsymbol{V}^a$ when $\omega = 0$ and becomes equal to identity when $\boldsymbol{V}^a = 0$.

From the equation

$$\boldsymbol{u}^\mu t_\mu = 1$$

it follows that

$$(\boldsymbol{u}^\mu) = \begin{pmatrix} 1 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix},$$

and from covariant constancy

$$(\boldsymbol{u}^\mu) = \begin{pmatrix} 1 \\ \vartheta^1 \cos\omega t - \vartheta^2 \sin\omega t - \omega y \\ \vartheta^2 \cos\omega t + \vartheta^1 \sin\omega t + \omega x \\ \vartheta^3 \end{pmatrix}, \qquad \vartheta^a = \text{const.}$$

From Eq. (9) with space rotation ($f = 0$)

$$\begin{pmatrix} \delta\xi^0 = \delta t \\ \delta\xi^1 = \delta x \\ \delta\xi^2 = \delta y \\ \delta\xi^3 = \delta z \end{pmatrix} = \begin{pmatrix} 0 \\ x + \varepsilon^1_2 y + \cos\omega t \varepsilon^1_3 z - \varepsilon^2_3 z \sin\omega t \\ y + \varepsilon^2_1 x + \cos\omega t \varepsilon^2_3 z + \sin\omega t \varepsilon^1_3 z \\ z + \varepsilon^3_1 x \cos\omega t + \sin\omega t \varepsilon^3_1 y - \sin\omega t \varepsilon^3_2 x + \cos\omega t \varepsilon^3_2 y \end{pmatrix}$$

(where $g_{ib}\varepsilon^b_k = \varepsilon_{ik} = -\varepsilon_{ki}$ are three infinitesimal rotation parameters) one gets

$$(\boldsymbol{u}^\mu) = \begin{pmatrix} 1 \\ -\omega y \\ \omega x \\ 0 \end{pmatrix}.$$

From Eq. (2) and $\eta = 0$ one gets

$$(g_{\mu\nu}) = \begin{pmatrix} \omega^2(x^2+y^2) & \omega y & -\omega x & o \\ \omega y & 1 & 0 & 0 \\ -\omega x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The above quantities fulfil all remaining Eq. (9) for all space–time symmetries with the *symmetric phase* $f(\upsilon)$ for Galilean transformation

$$(\partial_\mu f) = \begin{pmatrix} \omega y(V^1\cos\omega t - V^2\sin\omega t) - \omega x(V^2\cos\omega t + V^1\sin\omega t) \\ V^1\cos\omega t - V^2\sin\omega t \\ V^2\cos\omega t + V^1\sin\omega t \\ V^3 \end{pmatrix}.$$

The explicit form of the Schrödinger equation in rotating frame is

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}\vec{\nabla}^2\Psi + i\hbar\omega(y\partial_x - x\partial_y)\Psi,$$

in accordance with Białynicki-Birula *et al.* (1994), Białynicki-Birula and Białynicki-Birula (1997), Bordé *et al.* (1991), Kaliński *et al.* (1996), Lämmerzahl (1996), and Mashoon (1988).

APPENDIX (CONNECTION WITH THE WORK OF DUVAL AND KÜNZLE AND KUCHAŘ)

Identical quantities used by Duval and Künzle and in this paper are as follows:

Duval and Künzle		This paper
$(\gamma^{\mu\nu}, \psi_\mu, f)$	←——→	$(g^{\mu\nu}, t_\mu, f)$
$\boldsymbol{u}^\mu - \gamma^{\mu\nu}A_\nu$	←——→	$\boldsymbol{u}^\mu$
$\overset{u}{\gamma}_{\mu\nu} + A_\mu\psi_\nu + A_\nu\psi_\mu$	←——→	$g_{\mu\nu}$
$A_\mu\boldsymbol{u}^\mu - \frac{1}{2}\gamma^{\mu\nu}A_\mu A_\nu$	←——→	$\frac{1}{2}g_{\mu\nu}\boldsymbol{u}^\mu\boldsymbol{u}^\nu$

Gauge transformations of Duval and Künzle induce the gauge of Daŭtcourt's quantities and vice versa.

If the connection is Newtonian, then (where R denotes the curvature tensor)

$$\gamma^{\rho\lambda} R^{\mu}_{\nu\lambda\sigma} = R^{\mu\rho}_{\nu\sigma} = R^{\rho\mu}_{\sigma\nu},$$

which is equivalent to the last condition (8) in the paper of Duval and Künzle

$$\partial_{[\alpha} A_{\beta]} + \overset{u}{\gamma}_{[\alpha} \nabla_{\beta]} \boldsymbol{u}^{\lambda} = 0.$$

This is consistent with the gauge (11) and (12) (in the paper of Duval and Künzle, 1984)

$$\boldsymbol{u}^{\alpha} \to \hat{\boldsymbol{u}}^{\alpha} = \boldsymbol{u}^{\alpha} + g^{\alpha\lambda} w_{\lambda}, \qquad \chi \to \hat{\chi} = \chi + f,$$
$$A_{\alpha} \to \hat{A}_{\alpha} = A_{\alpha} + \partial_{\alpha} f + w_{\alpha} - \left(\boldsymbol{u}^{\lambda} w_{\lambda} + \frac{1}{2} g^{\mu\nu} w_{\mu} w_{\nu} \right)$$

if and only if

$$w_{\lambda} = \partial_{\lambda} w.$$

The explicit form of transformation of $\overset{u}{\gamma}_{\mu\nu}$—uniquely determined by (11) in Duval and Künzle (1984)—is

$$\overset{u}{\gamma}_{\mu\nu} \to \overset{u}{\gamma}_{\mu\nu} - \psi_{\mu}\, \partial_{\nu} w - \psi_{\nu}\, \partial_{\mu} w + \{2u^{\rho}\, \partial_{\rho} w + \gamma^{\rho\sigma}\, \partial_{\rho} w\, \partial_{\sigma} w\} \psi_{\mu} \psi_{\nu},$$

and gauge transformation of A_u uniquely determined by (11) and (8) is

$$A_{\mu} \to A_{\mu} + \partial_{\mu} f + \partial_{\mu} w - \left\{ \boldsymbol{u}^{\lambda}\, \partial_{\lambda} w + \frac{1}{2} \gamma^{\lambda\rho}\, \partial_{\lambda} w\, \partial_{\rho} w \right\} \psi_{\mu},$$

where u in the transformation formulas is that of Duval and Künzle. Condition imposed on A_{μ} can be defined in terms of Daŭtcourt's quantities $g_{\mu\nu}$ and $\boldsymbol{u}^{\mu}$ used in this paper. That is, condition (8) of Duval and Künzle (1984) has the form

$$g_{\lambda[\alpha} \nabla_{\beta]} \boldsymbol{u}^{\lambda} - \boldsymbol{u}^{\lambda} t_{[\alpha} \nabla_{\beta]} A_{\lambda} - A_{\lambda} t_{[\alpha} \nabla_{\beta]} \boldsymbol{u}^{\lambda} - g^{\lambda\sigma} A_{\lambda} t_{[\alpha} \nabla_{\beta]} A_{\sigma} = 0,$$

and *is identically fulfilled* as a consequence of the algebraic identity

$$A_{\mu} \boldsymbol{u}^{\mu} - \frac{1}{2} \gamma^{\mu\nu} A_{\mu} A_{\nu} \equiv \frac{1}{2} g_{\mu\nu} \boldsymbol{u}^{\mu} \boldsymbol{u}^{\nu},$$

where $\boldsymbol{u}$ *on the left hand side is that of Duval and Künzle but on the right is that of Daŭtcourt!*

This equation is covariant with respect to the gauge transformations. Gauge transformation of A_{μ} in terms of Daŭtcourt quantities is

$$A_{\mu} \to A_{\mu} + \partial_{\mu}(f + w) - \left\{ u^{\lambda}\, \partial_{\lambda} w + g^{\lambda\nu} A_{\lambda}\, \partial_{\nu} w + \frac{1}{2} g^{\lambda\nu}\, \partial_{\lambda} w\, \partial_{\nu} w \right\} t_{\mu}.$$

In the case the gauge f is symmetric (wave equation is invariant with respect to Galilei group) the covariant "γ" is invariant and

$$A_\mu = 0 \quad \text{and} \quad w = -f.$$

It should be stressed that in the general case, when the space–time is curved gauges f and w are independent, and covariant "γ" is not invariant, so covariant A cannot be determined in terms of Daŭtcourt quantities $\boldsymbol{u}$ and g in this way.

Quantities $g_{\mu\nu}$ and $\boldsymbol{u}^\mu$, which were used by Kuchař (1980), are equal to Daŭtcourt quantities used in this paper.

Kuchař found the explicit form of the wave equation (again up to some gauge transformations) in such coordinate systems, which have as one coordinate the absolute time t and the remaining coordinates lying in a simultaneity hyperplane (see Chap. VI and Eq. (6.8) of Kuchař, 1980). His Eq. (6.8) is in agreement with Białynicki-Birula *et al.* (1994), Białynicki-Birula and Białynicki-Birula (1997), Bordé *et al.* (1991), Kaliński *et al.* (1996), Lämmerzahl (1996), and Mashoon (1988) as well as Duval and Künzle (1984), but with the help of such special observer considerations (applied by Kuchař) it is difficult to obtain *generally* covariant equation.

REFERENCES

Białynicki-Birula, Z. and Białynicki-Birula, I. (1997). Rotational frequency shift. *Physical Review Letters* **78**, 2539.

Białynicki-Birula, I. *et al.* (1994). Lagrange equilibrium points in celestial mechanics and nonspreading wave packets for strongly driven Rydberg electrons. *Physical Review Letters* **73**, 1777.

Bordé Ch. J. *et al.* (1991). Optical Ramsey spectroscopy in a rotating frame: Sagnac effect in a matter–wave interferometer. *Physical Review Letters* **67**, 177.

Canarutto, D., Jadczyk, A., and Modugno, M. (1995). Quantum mechanics of a spin particle in a curved spacetime with absolute time. *Report On Mathematical Physics*, **36**, 95.

Courant, R. and Hilbert, D. (1961). *Methods of Mathematical Physics*, Wiley-Interscience, New York.

Daŭtcourt, G. (1990). On the Newtonian limit of general relativity. *Acta Physica Polon B* **21**, 755.

Duval, C. and Künzle, H. P. (1984). Minimal gravitational coupling in the Newtonian theory and the covariant Schrödinger equation. *General Relativity and Gravitation* **16**, 333.

Kaliński, M. *et al.* (1996). Trojan wave packets: Mathieu theory and generation from circular states. *Physical Review A* **53**, 1715. See also the other works concerning noninertial reference frames, not necessary rotating.

Kuchař, K. (1980). Gravitation, geometry, and nonrelativistic quantum theory. *Physical Review D* **22**, 1285.

Lämmerzahl, C. (1996). On the equivalence principle in quantum theory. *General Relativity and Gravitation*. **28**, 1043. See paragraph 5. See also references therein for *laser spectroscopy* experiments in the gravitational field.

Le Bellac, M. and Lévy-Leblond, J. M. (1973). Galilean electromagnetism. *Nuovo Cimento B* **14**, 217.

Lévy-Leblond, J. M. (1963). Galilei group and nonrelativistic quantum mechanics. *Journal of Mathematical Physics* **4**, 776.

Mashoon, B. (1988). Neutron interferometry in a rotating frame of reference. *Physical Review Letters* **61**, 2639.

Nijenhuis, A. (1952). *Theory of the Geometric Object* (*thesis*), University of Amsterdam, Amsterdam.
Schouten, J. A. (1951). *Tensor Analysis for Physicists*, Oxford University Press, Oxford.
Schrödinger, E. (1926). Quantisierung als Eigenwertproblem I. *Annalen der Physik* **79**, 361. See also Quantisierung als Eigenwertproblem II. *Annalen der Physik* **79**, 489.
Trautman, A. (1963). Sur la Théorie Newtonienne da la Gravitation. *Comptes Rendus de L'Academie des Sciences de Paris* **247**, 617.
Vitolo, R. (1999). Quantum structures in Galilei general relativity. *Annales de l'Institut Henri Poincare* **70**, 239.